

ON THE TOPOLOGY OF GERMS OF MEROMORPHIC FUNCTIONS AND ITS APPLICATIONS

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Germes of meromorphic functions has recently become an object of study in singularity theory. T. Suwa ([11]) described versal deformations of meromorphic germs. V.I. Arnold ([1]) classified meromorphic germs with respect to certain equivalence relations. The authors ([4]) started a study of topological properties of meromorphic germs. Some applications of the technique developed in [4] were described in [5] and [6].

In [4] the authors elaborated notions and technique which could be applied to compute such invariants of polynomials as Euler characteristics of fibres and zeta-functions of monodromy transformations associated with a polynomial (see [5]). Some crucial basic properties of the notions related to the topology of meromorphic germs were not discussed there. This has produced some lack of understanding of the general constructions. The aim of this note is to partially fill in this gap. At the same time we describe connections with some previous results and generalizations of them.

A polynomial P in $n + 1$ complex variables defines a map P from the affine complex space \mathbb{C}^{n+1} to the complex line \mathbb{C} . It is well known that the map P is a C^∞ -locally trivial fibration over the complement to a finite set in the line \mathbb{C} . The smallest of such sets is called the *bifurcation set* or the set of *atypical values* of the polynomial P . One is interested in describing the topology of the fibre of this fibration and its behaviour under monodromy transformations corresponding to loops around atypical values of the polynomial P . The monodromy transformation corresponding to a circle of big radius which contains all atypical values (the monodromy transformation of the polynomial P at infinity) is of particular interest.

The initial idea was to reduce calculation of the zeta-function of the monodromy transformation at infinity (and thus of the Euler characteristic of the generic fibre) of the polynomial P to local problems associated to different points at infinity, i.e., at the infinite hyperplane \mathbb{CP}_∞^n in the projective compactification \mathbb{CP}^{n+1} of the affine space \mathbb{C}^{n+1} . The possibility of such a localization for holomorphic germs was used in [3]. This localization can be expressed in terms of an integral with respect to the Euler characteristic, a notion introduced by the school of V.A. Rokhlin ([12]). However the results are not apply directly to a polynomial function since at a point of the infinite hyperplane \mathbb{CP}_∞^n a polynomial function defines not a

Key words and phrases. Germs of meromorphic functions, Milnor fibre, atypical values.

First author was partially supported by Iberdrola, INTAS-96-0713, RFBR 98-01-00612. Last two authors were partially supported by DGICYT PB97-0284-C02-01.

holomorphic but a meromorphic germ. Thus the idea of reducing calculations of the zeta-function of the monodromy transformation of a polynomial map at infinity to calculations of local zeta-functions corresponding to different points of the hyperplane \mathbb{CP}_∞^n is frustrated by the lack of such notions as the Milnor fibre, the monodromy transformation, ... for meromorphic germs. Therefore it was necessary to define corresponding invariants and to elaborate a technique for their calculation.

§1.- BASIC PROPERTIES

A *meromorphic germ* at the origin in the complex space \mathbb{C}^{n+1} is a ratio $f = \frac{P}{Q}$ of two holomorphic germs P and Q on $(\mathbb{C}^{n+1}, 0)$. For our purposes the following equivalence relation is appropriate. Two meromorphic germs $f = \frac{P}{Q}$ and $f' = \frac{P'}{Q'}$ are *equal* if and only if $P' = P \cdot U$ and $Q' = Q \cdot U$ for a holomorphic germ U not equal to zero at the origin: $U(0) \neq 0$.

A meromorphic germ $f = \frac{P}{Q}$ defines a map from the complement to the indeterminacy locus $\{P = Q = 0\}$ to the complex projective line \mathbb{CP}^1 . For any point $c \in \mathbb{CP}^1$, there exists $\varepsilon_0 > 0$ such that for any ε , $0 < \varepsilon \leq \varepsilon_0$, the map

$$f : B_\varepsilon \setminus \{P = Q = 0\} \rightarrow \mathbb{CP}^1$$

is a C^∞ locally trivial fibration over a punctured neighbourhood of the point c here B_ε denotes the closed ball of radius ε centred at the origin in \mathbb{C}^{n+1} (see [4]).

Definition. The fibre

$$\mathcal{M}_f^c = \{z \in B_\varepsilon : f(z) = \frac{P(z)}{Q(z)} = c'\}$$

of this fibration (for c' close to c enough) is called the *c-Milnor fibre* of the meromorphic germ f .

The Milnor fibre \mathcal{M}_f^c is a (non-compact) n -dimensional complex manifold with boundary.

Definition. The monodromy transformation of this fibration corresponding to a simple (small) loop around the value c is called the *c-monodromy transformation* of the meromorphic germ f .

Definition. A value $c \in \mathbb{CP}^1$ is called *typical* if the map $f : B_\varepsilon \setminus \{P = Q = 0\} \rightarrow \mathbb{CP}^1$ is a C^∞ locally trivial fibration over a neighbourhood of the point c (including c itself).

Notice that, if c is a typical value, the corresponding monodromy transformation is isotopic to the identity.

Theorem 1. *There exists a finite set $\Sigma \subset \mathbb{CP}^1$ such that for all $c \in \mathbb{CP}^1 \setminus \Sigma$ the c-Milnor fibres of f are diffeomorphic to each other and the c-monodromy transformations are trivial (i.e., isotopic to identity). In particular, the set of atypical values is finite.*

Proof. A resolution of the germ f is a modification of the space $(\mathbb{C}^{n+1}, 0)$ (i.e., a proper analytic map $\pi : \mathcal{X} \rightarrow \mathcal{U}$ of a smooth analytic manifold \mathcal{X} onto a neighbourhood \mathcal{U} of the origin in \mathbb{C}^{n+1} which is an isomorphism outside of a proper

analytic subspace in \mathcal{U}) such that the total transform $\pi^{-1}(H)$ of the hypersurface $H = \{P = 0\} \cup \{Q = 0\}$ is a normal crossing divisor at each point of the manifold \mathcal{X} . We assume that the map π is an isomorphism outside of the hypersurface H .

The fact that the preimage $\pi^{-1}(H)$ is a divisor with normal crossings implies that in a neighbourhood of any point of it, there exists a local system of coordinates y_0, y_1, \dots, y_n such that the liftings $\tilde{P} = P \circ \pi$ and $\tilde{Q} = Q \circ \pi$ of the functions P and Q to the space \mathcal{X} of the modification are equal to $u \cdot y_0^{k_0} y_1^{k_1} \cdots y_n^{k_n}$ and $v \cdot y_0^{l_0} y_1^{l_1} \cdots y_n^{l_n}$ respectively, where $u(0) \neq 0$ and $v(0) \neq 0$, k_i and l_i are nonnegative.

Remark 1. The values 0 and ∞ in the projective line \mathbb{CP}^1 are used as distinguished points for convenience: to have the usual notion of a resolution of a function for the numerator and for the denominator.

One can make additional blow-ups along intersections of pairs of irreducible components of the divisor $\pi^{-1}(H)$ so that the lifting $\tilde{f} = f \circ \pi = \frac{\tilde{P}}{\tilde{Q}}$ of the function f can be defined as a holomorphic map from the manifold \mathcal{X} to the complex projective line \mathbb{CP}^1 . This condition means that $\tilde{P} = V \cdot P'$, $\tilde{Q} = V \cdot Q'$ where V is a section of a line bundle, say \mathcal{L} , over \mathcal{X} , P' and Q' are sections of the line bundle \mathcal{L}^{-1} and P' and Q' have no common zeroes on \mathcal{X} . Let $\tilde{f}' = \frac{P'}{Q'}$.

On each component of the divisor $\pi^{-1}(H)$ and on all the intersections of several of them \tilde{f}' defines a map to the projective line \mathbb{CP}^1 . These maps have a finite number of critical values, say a_1, a_2, \dots, a_s .

Remark 2. If the function \tilde{f}' is constant on a component of a finite intersection of the irreducible divisors of $\pi^{-1}(H)$, then this constant value is critical. The value the function \tilde{f}' on an intersection of $n + 1$ components (this intersection is zero-dimensional) should also be considered as a critical value.

Let $c \in \mathbb{CP}^1$ be different from a_1, a_2, \dots, a_s . We shall show that for all c' from a neighbourhood of the point c (including c itself) the c' -Milnor fibres of the meromorphic function f are diffeomorphic to each other and the c' -monodromy transformations are trivial.

Let $r^2(z)$ be the square of the distance from the origin in the space \mathbb{C}^{n+1} and let $\tilde{r}^2(x) = r^2(\pi(x))$ be the lifting of this function to the space \mathcal{X} of the modification. In order to define the c' -Milnor fibre one has to choose $\varepsilon_0 > 0$ (the Milnor radius) small enough so that the level manifold $\{\tilde{r}^2(x) = \varepsilon^2\}$ is transversal to $\{\tilde{f}'(x) = c'\}$ for all ε such that $0 < \varepsilon \leq \varepsilon_0$. Let $\varepsilon_0 = \varepsilon_0(c)$ be the Milnor radius for the value c . Since $\{\tilde{f}'(x) = c\}$ is transversal to components of the divisor $\pi^{-1}(H)$ and to all their intersections, then also ε_0 is the Milnor radius for all c' from a neighbourhood of the point $c \in \mathbb{CP}^1$ (and the level manifold $\{\tilde{f}'(x) = c'\}$ is transversal to components of the divisor $\pi^{-1}(H)$ and to its intersections). This implies that for such c' the c' -Milnor fibres of the meromorphic germ f are diffeomorphic to each other and the c' -monodromy transformations are trivial. \square

Remark 3. The c -Milnor fibre for a generic value $c \in \mathbb{CP}^1$ can be called *the generic Milnor fibre* of the meromorphic germ f . One can easily see that the generic Milnor fibre of a meromorphic germ can be considered as embedded into the c -Milnor fibre for any value $c \in \mathbb{CP}^1$. Moreover the Euler characteristic of the generic Milnor fibre of a meromorphic germ is equal to zero and the zeta-function of the corresponding monodromy transformation (see [4]) is equal to $(1 - t)^0 - 1$.

§2.- ISOLATED SINGULARITIES AND EULER CHARACTERISTIC OF THE 0-MILNOR FIBRE

Let P be a polynomial in $n + 1$ complex variables. Suppose that the closure $\overline{V}_{t_0} \subset \mathbb{CP}^{n+1}$ of the level set $V_{t_0} = \{P = t_0\} \subset \mathbb{C}^{n+1}$ in the complex projective space $\mathbb{CP}^{n+1} \supset \mathbb{C}^{n+1}$ has only isolated singular points. Let A_1, \dots, A_r be those of them which lie in the affine space \mathbb{C}^{n+1} , and let B_1, \dots, B_s be those which lie in the infinite hyperplane \mathbb{CP}_∞^n . For t close enough to t_0 (and thus generic), the closure $\overline{V}_t \subset \mathbb{CP}^{n+1}$ of the level set $V_t = \{P = t\} \subset \mathbb{C}^{n+1}$ has no singular points in the space \mathbb{C}^{n+1} and may have isolated singularities only at the points B_1, \dots, B_s . It is known that

$$\chi(V_t) - \chi(V_{t_0}) = (-1)^{n+1} \left(\sum_{i=1}^r \mu_{A_i}(V_{t_0}) + \sum_{j=1}^s (\mu_{B_j}(\overline{V}_{t_0}) - \mu_{B_j}(\overline{V}_t)) \right). \quad (1)$$

We shall formulate a somewhat more general statement about meromorphic germs. This statement together with the formula for the difference of the Euler characteristics of the generic level set of a polynomial P and of a special one in terms of meromorphic germs defined by the polynomial P (see [6]) gives (1).

Theorem 2. *Let $f = \frac{P}{Q}$ be a germ of meromorphic function on the space $(\mathbb{C}^{n+1}, 0)$ such that the numerator P has an isolated critical point at the origin and, if $n = 1$, the germs of the curves $\{P = 0\}$ and $\{Q = 0\}$ have no common irreducible components. Then, for a generic $t \in \mathbb{C}$,*

$$\chi(\mathcal{M}_f^0) = (-1)^n (\mu(P, 0) - \mu(P + tQ, 0)).$$

Here $\mu(g, 0)$ stands for the usual Milnor number of the holomorphic germ g at the origin.

Proof. The Milnor fibre \mathcal{M}_f^0 of the meromorphic germ f has the following description. Let ε be small enough (and thus a Milnor radius for the holomorphic germ P). Then

$$\mathcal{M}_f^0 = B_\varepsilon(0) \cap (\{P + tQ = 0\} \setminus \{P = Q = 0\}),$$

for $t \neq 0$ with $|t|$ small enough (and thus t generic). Note that the zero-level set $\{P + tQ = 0\}$ is non-singular outside of the origin for $|t|$ small enough. The space $B_\varepsilon(0) \cap \{P = Q = 0\}$ is homeomorphic to a cone and therefore its Euler characteristic is equal to 1. Therefore

$$\chi(\mathcal{M}_f^0) = \chi(B_\varepsilon(0) \cap \{P + tQ = 0\}) - 1.$$

Now Theorem 2 is a consequence of the following well known fact (see, e.g., [2]).

Statement. Let $P : (\mathbb{C}^{n+1}, 0) \rightarrow (\mathbb{C}, 0)$ be a germ of a holomorphic function with an isolated critical point at the origin and let P_t be any deformation of P ($P_0 = P$). Let ε be small enough. Then for $|t|$ small enough

$$(-1)^n (\chi(B_\varepsilon(0) \cap \{P_t = 0\}) - 1)$$

is equal to the number of critical points of P (counted with multiplicities) which split from the zero level set, i.e., to

$$\mu(P, 0) - \sum_{Q \in \{P_t=0\} \cap B_\varepsilon} \mu(P_t, Q). \quad \square$$

Example 1. The following example shows the necessity of the condition that, for $n = 2$, the curves $\{P = 0\}$ and $\{Q = 0\}$ have no common components. Take $P = xy$ and $Q = x$.

Example 2. In Theorem 2 the difference of Milnor numbers (up to a sign) may be replaced by (the equal) difference of Euler characteristics of the corresponding Milnor fibres (of the germs P and $P + tQ$). However the formula obtained this way is not correct if the germ P has a nonisolated critical point at the origin. It is shown by the example $f = \frac{x^2 + z^2 y}{z^2}$.

The formula (1) is a direct consequence of Theorem 2 and the formula (2) of Theorem 2 from [6].

§3.- TOPOLOGICAL TRIVIALITY OF THE FAMILY $\{P + tQ\}$ AND TYPICAL VALUES OF MEROMORPHIC GERMS

Again let $f = \frac{P}{Q}$ be a meromorphic germ on $(\mathbb{C}^{n+1}, 0)$ such that the holomorphic germ P has an isolated critical point at the origin.

Theorem 3. *The value 0 is typical for the meromorphic germ f if and only if $\chi(\mathcal{M}_f^0) = 0$.*

Proof. “Only if” follows from the definition and Theorem 2.

“If” is a consequence of the result of A. Parusiński [8] (or rather of its proof). He has proved that, if $\mu(P) = \mu(P + tQ)$ for $|t|$ small enough, then the family of maps $P_t = P + tQ$ is topologically trivial. In particular the family of germs of hypersurfaces $\{P_t = 0\}$ is topologically trivial. For $n \neq 2$ this was proved by Lê D.T. and C.P. Ramanujam [7]. However in order to apply the result to the present situation it is necessary to have a topological trivialization of the family $\{P_t = 0\}$ which preserves the subset $\{P = Q = 0\}$ and is smooth outside the origin. For the family $P_t = P + tQ$, such a trivialization was explicitly constructed in [8] without any restriction on the dimension. \square

Example 3. If the germ of the function P has a non-isolated critical point at the origin then this characterization is no longer true. Take, for example, $P(x, y) = x^2 y^2$ and $Q(x, y) = x^4 + y^4$.

§4.- A GENERALIZATION OF THE PARUSIŃSKI–PRAGACZ FORMULA FOR THE EULER CHARACTERISTIC OF A SINGULAR HYPERSURFACE

Let X be a compact complex manifold and let \mathcal{L} be a holomorphic line bundle on X . Let s be a section of the bundle \mathcal{L} not identically equal to zero, and $Z := \{s = 0\}$ is its zero locus (a hypersurface in the manifold X). Let s' be another section of the bundle \mathcal{L} whose zero locus Z' is nonsingular and transversal to a Whitney stratification of the hypersurface Z . A. Parusiński and P. Pragacz have proved (see [9], Proposition 7) a statement which in terms of [6] can be written as follows

$$\chi(Z') - \chi(Z) = \int (\chi_x(Z) - 1) d\chi, \quad (2)$$

where $\chi_x(Z)$ is the Euler characteristic of the Milnor fibre of the germ of the section s at the point x (the definition of the integral with respect to the Euler characteristic can be found in [12] or [6]).

We shall indicate a more general formula which includes this as a particular case.

Theorem 4. *Let s be as above and let s' be a section of the bundle \mathcal{L} whose zero locus Z' is non-singular. Let f be the meromorphic function s/s' on the manifold X . Then*

$$\chi(Z') - \chi(Z) = \int_{Z \setminus Z'} (\chi_x(Z) - 1) d\chi + \int_{Z \cap Z'} \chi_{f,x}^0 d\chi, \quad (3)$$

where $\chi_{f,x}^0$ is the Euler characteristic of the 0-Milnor fibre of the meromorphic germ f at the point x .

Proof. Let F_t be the level set $\{f = t\}$ of the (global) meromorphic function f on the manifold X (with indeterminacy set $\{s = s' = 0\}$), i.e., $F_t = \{s - ts' = 0\} \setminus \{s = s' = 0\}$. By [6], for a generic value t one has

$$\chi(F_{gen}) - \chi(F_0) = \int_{F_0} (\chi_{f,x}^0(Z) - 1) d\chi + \int_{\{s=s'=0\}} \chi_{f,x}^0 d\chi,$$

where $\chi_{f,x}^0$ is the Euler characteristic of the 0-Milnor fibre of the meromorphic germ f at the point x . One has $F_0 = Z \setminus (Z \cap Z')$, $F_\infty = Z' \setminus (Z \cap Z')$ and in this case F_∞ is a generic level set of the meromorphic function f (since its closure is non-singular). Therefore $\chi(F_0) = \chi(Z) - \chi(Z \cap Z')$, $\chi(F_{gen}) = \chi(Z') - \chi(Z \cap Z')$. Finally, for $x \in F_0$, the germ of the function f at the point x is holomorphic and thus $\chi_{f,x}^0 = \chi_x(Z)$. \square

If the hypersurface Z' is transversal to all strata of a Whitney stratification of the hypersurface Z , then, for $x \in Z \cap Z'$, the Euler characteristic $\chi_{f,x}^0 = 0$ (Proposition 5.1 from [10]) and therefore the formula (3) reduces to (2).

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